SECTION – III

THE LEBESGUE INTEGRAL

Lebesgue integration is an alternative way of defining the integral in terms of measure theory that is used to inte grate a much broader class of functions than the Riemann integral or even the Riemann-Stieltjes integral. The idea behind the Lebesgue integral is that instead of approximating the total area by dividing it into vertical strips, one approximates the total area by dividing it into horizontal strips.

3.1 The shortcomings of the Riemann integral suggested the further investigations in the theory of integration. We give a resume of the Riemann Integral first.

Let f be a bounded real- valued function on the interval [a, b] and let

$a = \xi_0 < \xi_1 < \cdots < \xi_n = b$

Be a partition of [a, b]. Then for each partition we define the sums

$S = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) M_i$

and $s = \sum_{i=1}^{n} (\xi_i - \xi_{i-1}) m_i$

where

$M_i = \sup_{\xi_{i-1} < x < \xi_i} f(x)$, $m_i = \inf_{\xi_{i-1} < x < \xi_i} f(x)$

We then define the upper Riemann integral of f by

$R \int_a^b f(x)dx = \inf S$

With the infimum taken over all possible subdivisions of [a, b].

Similarly, we define the lower integral

$R \int_a^b f(x)dx = \sup s.$

The upper integral is always at least as large as the lower integral, and if the two are equal we say that f is Riemann integrable and call this common value the Riemann integral of f. We shall denote it by

$R \int_a^b f(x)$

To distinguish it from the Lebesgue integral, which we shall consider later.

By a **step function** we mean a function ψ which has the form

$$\psi(x) = c_i , \xi_{i-1} < x < \xi_i$$

for some subdivision of [a, b] and some set of constants c_i .

The integral of $\psi(x)$ is defined by

$$R\int_a^b \psi(x)\,dx = \sum_{i=1}^{n} c_i\left(\xi_i - \xi_{i-1}\right).$$

With this in mind we see that

$R\overline{\int_a^b} f(x)dx = \inf \int_a^b \psi(x)\,dx$

for all step function $\psi(x) \geq f(x)$.

Similarly,

$R\underline{\int_a^b} f(x)dx = \sup \int_a^b \phi(x)\,dx$

for all step functions $\phi(x) \leq f(x)$.

3.2. Example: If

$f(x) = \begin{cases} 1 \text{ if x is rational} \\ o \text{ if x irrational} \end{cases}$

then $R\overline{\int_a^b} f(x)dx = b - a$ and $R\underline{\int_a^b} f(x)dx = 0.$

Thus we see that f(x) is not integrable in the Riemann sense.

3.3. The Lebesgue Integral of a bounded function over a set of finite measure

The example we have cited just now shows some of shortcomings of the Riemann integral. In particular, we would like a function which is 1 in measurable set and zero elsewhere to be integrable and have its integral the measure of the set.

The function χ_E defined by

$$\chi_E = \begin{cases} 1 \in E \\ o\ x \notin E \end{cases}$$

is called the **characteristic function** on E. A linear combination

$$\phi(x) = \sum_{i=1}^{n} a_i\,\chi_E(x)$$

is called a **simple** function if the sets E_i are measurable. This representation for ϕ is not unique. However, we note that a function ϕ is simple if and only if it is measurable and assume only a finite number of values. If ϕ is simple function and $[a_1, a_2, \ldots, a_n]$ the set of non- zero values of ϕ, then

$\phi = \sum a_i \chi_{A_i}$,

where $A_i = \{\{x|\ \phi(x) = a_i\}$. **This representation for ϕ is called the canonical representation and it** is characterized by the fact that the A_i are disjoint and the a_i distinct and non- zero.

If ϕ vanishes outside a set of finite measure, we define the integral ϕ by

$$\int \phi(x)dx = \sum_{i=1}^{n} a_i mA_i$$

when ϕ has the canonical representation $\phi = \sum_{i=1}^{n} a_i\chi_{A_i}$. we sometimes abbreviate the expression for this integral to $\int \phi$. If E is any measurable set, we define $\int_E \emptyset = \int \phi.\chi_E$

It is often convenient to use representations which are not canonical, and the following lemma is useful.

3.4. Lemma. If $E_1, E_2, \ldots, E_n$ are disjoint measurable subset of E then every linear combination

$\phi = \sum_{i=1}^{n} c_i\chi_{E_i}$

With real coefficients $c_1, c_2, \ldots, c_n$ is a simple function and

$\int \phi = \sum_{i=1}^{n} c_i mE_i$.

Proof. It is clear that ϕ is a simple function. Let $a_1, a_2, \ldots, a_n$ denote the non- zero real number in $\phi(E)$. For each j = 1, 2, …, n . Let

$$A_j = \bigcup_{c_i = a_j} E_i$$

Then we have $A_j = \phi^{-1}(a_j) = \{x | \phi(x) = a_j\}$

and the canonical representation

$$\phi = \sum_{j=1}^{n} a_j\chi_{A_j}$$

Consequently, we obtain

$$\int \phi = \sum_{j=1}^{n} a_j mA_j$$

$$= \sum_{j=1}^{n} a_j m \quad [\cup_{c_i=a_j} E_i]$$

$$= \sum_{j=1}^{n} a_j \sum_{c_i=a_j}^{n} mE_i$$ (Since E_i are disjoint, additivity of measures applies)

$$\sum_{j=1}^{n} c_j mE_i$$

This completes the proof of the theorem.

3.5. **Theorem.** Let ϕ and ψ be simple functions which vanish outside a set of finite measure. Then

$\int (a\phi + b\psi) = a \int \phi + b \int \psi$ and, if $\phi \geq \psi$ a.e., then $\int \phi \geq \int \psi$

Proof. Let $\{A_i\}$ and $\{B_i\}$ be the sets which occur in the canonical representations of ϕ and ψ. Let A_0 and B_0 be the sets where ϕ and ψ are zero. Then the sets E_k obtained by taking all the intersection $A_i \cap B_j$ form a finite disjoint collection of measurable sets, and we write

$$\phi = \sum_{k=1}^{N} a_k\chi_{E_k}$$

$$\psi = \sum_{k=1}^{N} b_k \chi_{E_k}$$

and so

$$\begin{aligned} a\phi + b\psi &= a\sum_{k=1}^{N} a_k \chi_{E_k} + b\sum_{k=1}^{N} b_k \chi_{E_k} \\ &= \sum_{k=1}^{N} aa_k \chi_{E_k} + \sum_{k=1}^{N} b_k \chi_{E_k} \\ &= \sum_{k=1}^{N} (aa_k + bb_k)\chi_{E_k} \end{aligned}$$

Therefore

$$\begin{aligned} a\phi + b\psi &= \sum_{k=1}^{N} (aa_k + bb_k) mE_k \\ &= a\sum_{k=1}^{N} a_k mE_k + b\sum_{k=1}^{N} b_k mE_k \\ &= a\int \phi + b\int \psi\,. \end{aligned}$$

To prove the second statement, we note that

$$\int \phi - \int \psi = \int \phi - \psi \ \geq 0\,,$$

Since the integral of **a simple function which is greater than or equal to zero almost everywhere is non- negative** by the definition of the integral.

3.6. Remark. We know that for any simple function ϕ we have

$$\phi = \sum_{k=1}^{N} a_i \chi_{E_i}$$

Suppose that this representation is neither canonical nor the sets E_i's are disjoint. Then using the fact that characteristics functions are always simple function we observe that

$$\begin{aligned} \int \phi &= \int a_1\chi_{E_1} + \int a_2\chi_{E_2} + \ldots + \int a_n\chi_{E_n} \\ &= a_{1\int \chi_{E_1}} + a_{2\int \chi_{E_2}} + \cdots + a_{n\int \chi_{E_n}} \\ &= a_1 mE_1 + a_2 mE_2 + \cdots + a_n mE_n \end{aligned}$$

$$= \sum_{k=1}^{N} a_i mE_i$$

Hence for any representation of ϕ, we have

$$\int \phi = \sum_{k=1}^{N} a_i mE_i$$

Let f be a bounded real valued function and E be a measurable set of finite measure. By analogy with the Riemann integral we consider for simple functions ϕ and ψ the numbers

$$\inf_{\psi \geq f} \int_E \psi$$

and

$$\sup_{\phi \leq f} \int_E \phi$$

and ask when **these two numbers are equal.** The answer is given by the following proposition .

3.7. Theorem. Let f be defined and bounded on a measurable set E with mE finite. In order that

$$\inf_{f \leq \psi} \int_E \psi\,(x)dx = \sup_{f \geq \psi} \int_E \psi(x)dx$$

For all simple functions ϕ and ψ, it is necessary and sufficient that f be measurable.

Proof. Let f be bounded by M and suppose that f is measurable. Then the sets

$$E_k = \left\{x \middle| \frac{KM}{n} \geq f(x) > \frac{(K-1)M}{n}\right\}, -n \leq K \leq n\,,$$

Are measurable, disjoint and have union E. Thus

$$\sum_{k=-n}^{n} mE_k = mE$$

The simple function defined by

$$\psi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} k\chi_{E_k}(x)$$

and

$$\phi_n(x) = \frac{M}{n} \sum_{k=-n}^{n} (k-1)\chi_{E_k}(x)$$

satisfy

$$\phi_n(x) \leq f(x) \leq \psi_n(x)$$

Thus $\qquad \inf \int_E \psi(x)dx \le \int_E \psi_n(x)dx = \frac{M}{n}\sum_{k=-n}^{n} kmE_k$

and $\qquad \sup \int_E \phi(x)dx \ge \int_E \phi_n(x)dx = \frac{M}{n}\sum_{k=-n}^{n}(k-1)m\,E_k$

hence $\quad 0 \le \inf \int_E \psi(x)dx - \sup \int_E \phi(x)dx \le \frac{M}{n}\sum_{k=-n}^{n} mE_k = \frac{M}{n} mE$.

Since n is arbitrary we have

$$\inf \int_E \psi(x)dx - \sup \int_E \phi(x)dx = 0\,,$$

and the condition is sufficient.

Suppose now that $\inf_{\psi\ge f} \int_E \psi(x)dx = \sup_{\phi\le f} \int_E \phi(x)dx$.

Then given n there are simple functions ϕ_n and ψ_n such that

$$\phi_n(x) \le f(x) \le \psi_n(x)$$

And $\quad (1) \qquad \int \psi_n(x)dx - \int \phi_n(x)dx < \frac{1}{n}$

Then the functions $\qquad \psi^* = \inf \psi_n$

And $\qquad \phi^* = \sup \phi_n$

Are measurable and $\qquad \phi^*(x) \le f(x) \le \psi^*(x)$.

Now the set

$$\Delta = \{x \mid \phi^*(x) < \psi^*(x)\}$$

is the union of the sets

$$\Delta_v = \left\{x \,\middle|\, \phi^*(x) < \psi^*(x) - \frac{1}{v}\right\}.$$

But each Δ_v is contained in the set $\left\{x \middle| \phi_n(x) < \psi_n(x) - \frac{1}{v}\right\}$, and this latter set by (1) has measure less than $\frac{v}{n}$. Since n is arbitrary, $m\Delta_v = 0$ and so $m\Delta = 0$. Thus $\phi^* = \psi^*$ except on a set of measure zero, and $\phi^* = f$ except on a set of measure zero. Thus f is measurable and the condition is also necessary.

3.8. **Definition.** If f is a bounded measurable function defined on a measurable set E with mE finite, we define the Lebesgue integral of f over E by

$$\int_E f(x)dx = \inf \int_E \psi(x)$$

for all simple functions $\psi \ge f$.

By previous theorem, this may also be defined as

$$\int_E f(x)dx = \sup \int_E \phi(x)$$

for all simple functions $\phi \leq f$.

We sometime write the integral as $\int_E f$. If E =[a,b] we write $\int_a^b f$ instead of $\int_{[a,b]} f$.

Definition and existence of the Lebesgue integral for bounded functions

3.9. Definition. Let F be a bounded function on E and let E_k be a subset of E. Then we define $M[f, E_k]$ and $m[f, E_k]$ as

$$M[f, E_k] = \underset{x\varepsilon E_k}{\text{l.u.b}}\, f(x)$$

$$m[f, E_k] = \underset{x\varepsilon E_k}{\text{g.l.b}}\, f(x)$$

3.10. Definition. By a **measurable partition of E we mean a finite collection** $P = \{E_1, E_2, \ldots, E_n\}$ of measurable subsets of E such that

$$\bigcup_{k=1}^{n} E_k = E$$

And such that $m(E_j \cap E_k) = 0$ $(j, k = 1,2, \ldots, n, j \neq k)$

The sets $E_1, E_2, \ldots, E_n$ are called the **components of P.**

If P and Q are measurable partitions, then Q is called a refinement of P if every component of Q is wholly contained in some component of P.

Thus a measurable partition P is a finite collection of subsets whose union is all of E and whose intersections with one another have measure zero.

3.11. Definition. Let f be a bounded function on E and let $P=\{E_1, E_2, \ldots, E_n\}$ be any measurable partition E. we define the upper sum U[f, P] as

$$U[f;\ P] = \sum_{k=1}^{n} M[f; E_k].\, mE_k$$

Similarly, we define the lower sum L[f; P] as

$$L[f;\ P] = \sum_{k=1}^{n} m[f; E_k].\, mE_k$$

As in the case of Riemann integral, we can see that every upper sum for f is greater than or equal to every lower sum for f.

We then define the Lebesgue upper and lower integral of a bounded function f on E by

$$\inf_P U[f;\ P] \text{ and } \sup_P L[f;\ P]$$

Respectively taken over all measurable position of E. We denote them respectively by

$$\overline{\int_E} f \text{ and } \underline{\int_E} f$$

3.12. Definition. We say that a bounded function f on E is Lebesgue integrable on E if

$$\overline{\int_E} f \text{ and } \underline{\int_E} f$$

Also we know that if ψ is a simple function, then

$$\int_E \psi = \sum_{k=1}^{n} a_k mE_k$$

Keeping this in mind, we see that

$$\overline{\int_E} f = \inf \overline{\int_E} \psi(x)dx$$

For all simple functions $\psi(x) \geq f(x)$. Similarly

$$\underline{\int_E} f = \sup \underline{\int_E} \phi(x)dx$$

For all simple functions $\phi(x) \leq f(x)$.

Now we use the theorem :

" Let f be defined and bounded on a measurable set E with mE finite. In order that

$$\inf_{f\leq\psi} \int_E \psi(x)dx = \sup_{f\geq\phi} \int_E \phi(x)dx$$

for all simple functions ϕ and ψ, it is necessary and sufficient that f is measurable."

And our definition of **Lebesgue integration** takes the form :

" If f is a bounded measurable function defined on a measurable set E with mE finite , we define the (Lebesgue) integral of f over E by

$$\int_E f(x)dx = \inf \int_E \psi(x)dx$$

for all simple functions $\psi \geq f$."

The following theorem shows that the Lebesgue integral is in fact a generalization of the Riemann integral.

3.13. Theorem. Let f be a bounded function defined on [a,b]. If f is Riemann integrable on [a, b], then it is measurable and

$$R\int_a^b f(x)dx = \int_a^b f(x)$$

Proof . Since f is a bounded function defined on [a, b] and is Riemann integrable, therefore,

$$R\int_a^{\overline{b}} f(x)dx = \inf_{\phi\geq f}\int_a^b \phi(x)dx$$

and

$$R\int_{\underline{a}}^b f(x)dx = \sup_{\psi\leq f}\int_a^b \psi(x)dx$$

for all step functions ϕ and ψ and then

$$R\int_a^{\overline{b}} f(x)dx = R\int_{\underline{a}}^b f(x)dx$$

$$\Rightarrow \inf_{\phi\geq f}\int_a^b \phi(x)dx = \sup_{\psi\leq f}\int_a^b \psi(x)dx \qquad \text{(i)}$$

Since every step function is a simple function, we have

$$R\int_{\underline{a}}^b f(x)dx = \sup_{\psi\leq f}\int_a^b \psi(x)dx \leq \inf_{\phi\geq f}\int_a^b \phi(x)dx = R\int_a^{\overline{b}} f(x)dx$$

Then (i) implies that

$$\sup_{\psi\leq f}\int_a^b \psi(x)dx = \inf_{\phi\geq f}\int_a^b \phi(x)dx$$

and this implies that f is measurable also.

3.14. Comparison of Lebesgue and Riemann integration

(1) The most obvious difference is that in Lebesgue's definition we divide up the interval into subsets while in the case of Riemann we divide it into subintervals.

(2) In both Riemann's and Lebesgue's definitions we have upper and lower sums which tend to limits. In Riemann case the two integrals are not necessarily the same and the function is integrable only if they are same. In the Lebesgue case the two integrals are necessarily the same, their equality being consequence of the assumption that the function is measurable.

(3) Lebesgue's definition is more general than Riemann. We know that if function is the R- integrable then it is Lebesgue integrable also, but the converse need not be true. For example the characteristic function of the set of irrational points have Lebesgue integral but is not R- integrable.

Let χ be the characteristic function of the irrational numbers in [0,1]. Let E_1 be the set of irrational number in [0,1], and let E_2 be the set of rational number in [0,1]. Then $P = [E_1, E_2]$ is a measurable partition of (0,1]. Moreover, χ is identically 1 on E_1 and χ is identically 0 on E_2 . Hence $M[\chi, E_1] = m[\chi, E_2] = 1$, while $M[\chi, E_1] = m[\chi, E_2] = 0$. Hence $U[\chi, P] = 1.m\,E_1 + 0.m\,E_2 = 1$. Similarly $L[\chi, P] = 1.m\,E_1 + 0.M\,E_2 = 1$. Therefore, $U[\chi, P] = L[\chi, P]$.

For Riemann integration

$M[\chi,J] = 1$, $m[\chi,J] = 0$

for any interval $J \subset [0,1]$

$$\therefore U[\chi, J] = 1, L[\chi, J] = 0 .$$

$\therefore$ The function is not Riemann- integrable.

3.15. Theorem. If f and g are bounded measurable functions defined on a set E of finite measure, then

(i) $\int_E af = a\int_E f$

(ii) $\int_E (f+g) = \int_E f + \int_E g$

(iii) If $f \le g$ a.e., then $\int_E f \le \int_E g$

(iv) If $f = g$ a.e., then $\int_E f = \int_E g$

(v) If $A \le f(x) \le B$, then $AmE \le \int_E f \le BmE.$

(vi) If A and B are disjoint measurable set of finite measure, then $\int_{A\cup B} f = \int_A g + \int_B f$

Proof. We know that if ψ is a simple function then so is a ψ.

Hence $\int_E af = \inf_{\psi\ge f}\int_E a\psi = a\inf_{\psi\ge f}\int_E \psi = a\int_E f$

Which proves (i).

To prove (ii) let ε denote any positive real number. These are simple functions $\phi \le f, \psi \ge f, \xi \le g$ and $\eta \ge g$ satisfying

$$\int_E \phi(x)dx > \int_E f - \varepsilon, \qquad \int_E \psi(x)dx < \int_E f + \varepsilon,$$

$$\int_E \xi(x)dx > \int_E g - \varepsilon, \qquad \int_E \eta(x) < \int_E g + \varepsilon,$$

Since $\phi + \xi \le f + g \le \psi + \eta$, we have

$$\int_E (f+g) \ge \int_E (\phi + \xi) = \int_E \phi + \int_E \xi > \int_E f + \int_E g - 2\varepsilon$$

$$\int_E (f+g) \le \int_E (\psi + \eta) = \int_E \psi + \int_E \eta < \int_E f + \int_E g + 2\varepsilon$$

Since these hold for every $\varepsilon > 0$, we have

$$\int_E (f+g) = \int_E f + \int_E g$$

To prove (iii) it suffices to establish

$$\int_E g - f \geq 0$$

For every simple function $\psi \geq g - f$, we have $\psi \geq 0$ almost everywhere in E. This means that $\int_E \psi \geq 0$

Hence we obtain

$$\int_E (g-f) = \inf_{\psi \geq (g-f)} \int_E \psi(x) \geq 0 \qquad (1)$$

Which establishes (iii).

Similarly we can show that

$$\int_E (g-f) = \sup_{\psi \leq (g-f)} \int_E \psi(x) \leq 0 \qquad (2)$$

Therefore, from (1) and (2) the result (iv) follows.

To prove (v) we are given that

$$A \leq f(x) \leq B$$

Applying (iv) we are given that

$$\int_E f(x)dx \leq \int_E Bdx = B\int_E dx = BmE$$

That is, $\int_E f \leq BmE$

Similarly we can prove that $\int_E f \geq BmE.$

Now we prove (vi).

We know that $\chi_{A\cup B} = \chi_A + \chi_B$

Therefore , $\int_{A\cup B} f = \int_{A\cup B} \chi_{A\cup B} f = \int_{A\cup B} f(\chi_A + \chi_B)$

$$= \int_{A\cup B} f\chi_A + \int_{A\cup B} f\chi_B$$

$$= \int_A f + \int_B f$$

Which proves the theorem.

3.16. Corollary. If f and g are bounded measurable function then

If $f(x) \geq 0$ on E then $\int_E f \geq 0$ and

If $f(x) \leq 0$ on E then $\int_E f \leq 0$.

Proof : Let ψ be a simple function such that $\psi \geq f$

Since $f(x) \geq 0$ on E $\Rightarrow \psi \geq 0$ on E

$\Rightarrow \int_E \psi \geq 0 \Rightarrow \underset{\psi \geq f}{\text{Inf}} \int_E \psi \geq 0$ i.e. $\int_E f \geq 0$

Similarly, Let ϕ be a simple function such that $\phi \leq f$. Since $f(x) \geq 0$ on E

$\Rightarrow \phi \leq 0$ on E

$\Rightarrow \int_E \phi \leq 0 \Rightarrow \underset{\phi \leq f}{\text{Sup}} \int_E \phi \leq 0$ i.e. $\int_E f \leq 0$

3.17. Corollary. If m(E) = 0 , then $\int_E f = 0$

Or

Integrals over set of measure zero are zero.

Proof : Since f is bounded on E so there exist constant A and B such that

$$A \leq f(x) \leq B$$

$$\Rightarrow A.m(E) \leq \int_E f(x)dx \leq B.m(E) \quad \forall x \in E$$

Since m(E) = 0 $\Rightarrow \int_E f = 0$

3.18. Corollary. If f(x) = k a.e. on E then $\int_E f = k.m(E)$. In particular if f = 0 a.e. on E then $\int_E f = 0$

Proof : Since f(x) = k a.e on E then $\int_E f = 0$

3.19. Corollary. If f = g a.e then $\int_E f = \int_E g$ but converse is not true.

Proof : consider the functions

$f : [-1,1] \longrightarrow R$ and $g : [1,1] \longrightarrow R$

as $f(x) = \begin{cases} 2 \text{ if } x \leq 0 \\ 0 \text{ if } x > 0 \end{cases}$ and g(x) =1 $\forall$ x

Clearly f and g are bounded and measurable functions.

$\Rightarrow$ f and g are lebesgue integrable on [-1,1]

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx$$

$$= \int_{-1}^{0} 2dx + \int_{0}^{1} 0.dx = 2$$

$$\int_{-1}^{1} g(x) = \int_{-1}^{1} 1dx = 1.m([-1,1]) = 1.2 = 2$$

Therefore $\int_E f = \int_E g$

But f ≠ g a.e on [-1,1]

$$\{ \because m\{x \in [-1,1] \, ; f \neq g \} = 2 \neq 0 \}$$

Therefore f ≠ g a.e on E

f and g are not equal even at a single point of [-1,1] as these are defined.

3.20. Corollary. If f = 0 a.e on E then $\int_E f = 0$ but converse is not true.

Proof : Consider the function f : [-1,1] → R as $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx$$

$$= -1 + 1 = 0$$

Clearly f ≠ 0 a.e as $m\{x \in [-1,1] \, ; f \neq 0 \} = m[-1,1] = 2 \neq 0$

So converse is not true.

3.21.Corollary. If $\int_E f = 0$ and $f \geq 0$ on E then f = 0 a.e.

Proof : Suppose E has a subset A where f(x) > 0,

i.e. $A = \bigcup_{x=1}^{\infty} \left\{x \in E \, ; f(x) > \frac{1}{n}\right\}$

Let $E_1(n) = \left\{ x \in E \; ; f(x) > \frac{1}{n}\right\}$

If possible, suppose there is a positive integer N such that $m(E_1(N)) > 0$.

Then $\int_E f \geq \int_{E_1(N)} f \geq \frac{1}{N} m(E_1(N)) > 0$

Which contradicts the fact that $\int_E f = 0$

Thus, $m(E_1(n)) = 0$ for all $n \geq 1$.

This proves the corollary.

3.22. Corollary. Let f be a bounded measurable function on a set of finite measure E. Then

$\left|\int_E f\right| \leq \int_E |f|$

Proof : The function |f| is measurable and bounded

Now $-|f| \leq f \leq |f|$ on E

By the linearity and monotonicity of integration,

$-\left|\int_E f\right| \leq \int_E f \leq \int_E |f|$

$\Rightarrow \left|\int_E f\right| \leq \int_E |f|$

3.23. The Monotone Convergence Theorem

Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions on E. If $\{f_n\} \rightarrow f$ pointwise a.e on E, then $\lim_{n\to\infty} \int_E f_n = \int_E f$

Proof : Since $\{f_n\}$ is an increasing sequence

So $f_n \leq f$ a.e $\forall$ n

$\Rightarrow \overline{\lim} \int f_n \leq \int f$ … (1)

Now by Fatou's Lemma $\int f \leq \underline{\lim} \int f_n$ … (2)

From (1) and (2), we have

$$\overline{\lim} \int f_n = \underline{\lim} \int f$$

Hence the result .

Case II If f is a bounded function on E, then theorem is trivially true. Since in this case

$|f(x)| \leq M \ \forall\, x \in E$ for some number M and thus $\in > 0$, one can choose a $\delta = \left(\frac{\in}{M}\right) > 0$ for which m(A) $< \delta$, then $\int_A f \leq M \int_A 1 = M.\,m(A) < \in$.

3.24. Remark : The technique used in above theorem helps us to evaluate the lebesgue integral of non-negative bounded and unbounded functions.

3.25. Example : Evaluate the Lebesgue integral of the function f : [0,1] → R

$$f(x) = \begin{cases} 1/x^{1/3} & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0 \end{cases}$$

Clearly f is unbounded, non-negative function defined on [0,1]. Now define a sequence of functions $\{f_n\}$ on [0,1] as

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } n < f(x) \end{cases}$$

$$\text{i.e. } f_n(x) = \begin{cases} f(x) & \text{if } x \geq \frac{1}{n^3} \\ n & \text{if } x < \frac{1}{n^3} \end{cases}$$

Clearly $\{f_n\}$ is increasing sequence of non-negative measurable functions such that $f_n \rightarrow f$. So by monotone convergence theorem

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\int_0^1 f_n(x)dx$$

$$= \lim_{n\to\infty}\left[\int_0^{1/n^3} f_n(x)dx + \int_{1/n^3}^1 f_n(x)dx\right]$$

$$= \lim_{n\to\infty}\left[\int_0^{1/n^3} ndx + \int_{1/n^3}^1 f(x)dx\right]$$

$$= \lim_{n\to\infty}\left[[nx]_0^{1/n^3} + \int_{1/n^3}^1 x^{-1/3}dx\right]$$

$$= \lim_{n\to\infty}\left[n.\frac{1}{n^3} + \frac{3}{2}\left(1-\frac{1}{n^2}\right)\right]$$

$$= 0 + \frac{3}{2} = \frac{3}{2}$$

3.26. Theorem(Lebesgue Bounded Convergence Theorem). Let $< f_n >$ be a sequence of measurable functions defined on a set E of finite measure and suppose that $< f_n >$ is uniformly bounded, that is , there exist a real number M such that $|f_n(x)| \le M$ for all n ε N and for all x ε E . If $\lim_{n\to\infty} f_n(x) = f(x)$ for each x in E, then

$$\int_E f = \lim_{n\to\infty}\int_E f_n \ .$$

Proof. We shall apply Egoroff's theorem to prove this theorem. Accordingly for a given $\varepsilon > 0$, there is an N and a measurable set $E_0 \subset E$ such that $mE_0^c < \frac{\epsilon}{4M}$ and for $n \ge N$ and x ε E_0 we have

$$|f_n(x) - f(x)| < \frac{\epsilon}{2m(E)}$$

$$\left|\int_E f_n - \int_E f\right| = \left|\int_E (f_n - f)\right| \le \int_E |f_n - f|$$

$$= \int_{E_0} |f_n - f| + \int_{E_0^c} |f_n - f|$$

$$< \frac{\epsilon}{2m(E)}.m(E_0) + \frac{\epsilon}{4M}2M$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Hence

$$\int_E f_n \to \int_E f$$

3.27. Remark : Bounded Convergence Theorem need not be true in Riemann integral .

3.28. Example : Let $\{r_i\}$ be a sequence of all rational numbers in [0,1].

Define $S_n = \{r_i : i = 1,2,\dots,n\}, n \in N$

and for each $n \in N$, consider the function $f_n(x) = \begin{cases} 1 \text{ if } x \in S_n \\ 0 \text{ if } x \notin S_n \end{cases} = \{r_1, r_2, \dots, r_n\}$

clearly each f_n is bounded, also f_n is discontinuous at n-points in [0,1] namely points of S_n i.e., $r_1, r_2, \dots, r_n$.

At $x = r_1$

$$\lim_{x \to r_1^-} f_n(x) \neq f_n(r_1) \neq \lim_{x \to r_1^+} f_n(x)$$

Hence Riemann integrable on [0,1]

$[\because \text{A function is Riemann integrable, if it is continous except at a finite number of discontinuity}]$

Now we have proved that

$$\lim_{n\to\infty} R\int f_n(x)dx \neq R\int_0^1 \lim_{n\to\infty} f_n(x)dx$$

$$\Rightarrow R\int_0^1 f_n(x)dx = \int_0^1 f_n(x)dx = \int_{S_n \cup S_n^c} f_n(x)dx$$

$$\{\because S_n \cup S_n^c = [0,1]\}$$

$$= \int_{S_n} f_n(x)dx + \int_{S_n^c} f_n(x)dx$$

$$= \int_{S_n} 1\,dx + \int_{S_n^c} 0\,dx = 1.m(S_n) = 0$$

$$[\because \{S_n\} \text{sequence of rationals } m(S_n) = 0]$$

$$\Rightarrow \lim_{n\to\infty} R\int_0^1 f_n(x)dx = 0$$

Clearly $\{f_n\}$ is convergent to f when f is defined as

$$f(x) = \begin{cases} 1 \text{ if f is rational in } [0,1] \\ 0 \text{ if f is irrational in } [0,1] \end{cases}$$

and f is not Riemann – integrable on [0,1]

$$\Rightarrow R\int_0^1 f(x)dx \text{ does not exists .}$$

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq R \int_0^1 \lim_{n\to\infty} f_n(x)dx$$

So bounded convergence theorem does not hold in Riemann integral .

The integral of a non-negative function

3.29. Definition. If f is a non-negative measurable function defined on a measurable set E, we define

$$\int_E f = \sup_{h\leq f} \int_E h\,,$$

Where h is a bounded measurable function such that $m\{x|h(x) \neq 0\}$ is finite.

3.30. Theorem. If f and g are non-negative measurable functions, then

(i) $\int_E cf = c\int_E f > 0$

(ii) $\int_E (f+g) = \int_E f + \int_E g$ and

(iii) $\text{If } f \leq g \text{ a.e.}, \text{then}$

$$\int_E f \leq \int_E g$$

Proof. The proof of (i) and (iii) follow directly from the theorem concerning properties of the integrals of bdd functions.

We prove (ii) in detail.

If $h(x) \leq f(x)$ and $k(x) \leq g(x)$, we have $h(x) + k(x) \leq f(x) + g(x)$, and so

$$\int_E (h+k) \leq \int_E (f+g)$$

i.e. $\int_E h + \int_E k \leq \int_E (f+g)$.

Taking suprema, we have

(iv) $\int_E f + \int_E g \leq \int_E (f+g)$

On the other hand, let ℓ **be a bounded measurable function** which vanishes outside a set finite measure and which is not greater than $(f+g)$. Then we define the functions h and k by setting

$$h(x) = \min(f(x), \ell(x))$$

and $$k(x) = \ell(x) - h(x)$$

we have $$h(x) \leq f(x),$$

$$k(x) \leq g(x)$$

while h and k are bounded by the bound ℓ and vanish where ℓ vanishes. Hence

$$\int_E \ell = \int_E h + \int_E k \le \int_E f + \int_E g$$

And so taking supremum, we have

$$\sup_{\ell \le f+g} \int_E \ell \le \int_E f + \int_E g$$

That is,

(v) $\int_E f + \int_E g \ge \int_E (f+g)$

From (iv) and (v), we have

$$\int_E (f+g) = \int_E f + \int_E g$$

3.31. Fatou's lemma. If $< f_n >$is a sequence of non-negative measurable functions and $f_n(x) \to f(x)$ almost everywhere on a set E, then

$$\int_E f \le \underline{\lim} \int_E f_n$$

Proof. Let h be a bounded measurable function which is not greater than f and which vanishes outside a set E' of finite measure. Define a function h_n by setting

$h_n(x) = \min\{h(x), f_n(x)\}$

Then h_n is bounded but bounds for h and vanishes outside E' . Now $h_n(x) \to h(x)$ for each x in E' .

Therefore by "**Bounded Convergence theorem"** we have

$$\int_E h = \int_{E'} h = \lim \int_{E'} h_n \le \underline{\lim} \int_E f_n$$

Taking the supremum over h, we get

$$\int_E f \le \underline{\lim} \int_E f_n$$

3.32. The inequality in Fatou's lemma may be strict

Consider a sequence $\{f_n\}$ defined on R as

$$f_n(x) = \begin{cases} 1 \text{ if } x \in [n, n+1] & E_1 \\ 0 \quad \text{otherwise} & E_2 \end{cases}$$

Clearly sequence $\{f_n\}$ is sequence of non – negative measurable funcyions defined on R and

$\lim_{n\to\infty} f_n = f$ where $f = 0 \Rightarrow \int_R f = 0$

Also $\int_R f_n = \int_{E_1 \cup E_2} f_n = \int_{E_1} f_n + \int_{E_2} f_n$

$= \int_{E_1} 1 + 0 = m(E_1) = 1$

$\Rightarrow \underline{\lim} \int_R f_n = 1$ and we know that $0 < 1$

So $\int_R f < \underline{\lim} \int_R f_n$

3.33. Fatou's lemma need not good unless the function f_n is non – negative

Let us consider the function $f_n(x) = \begin{cases} -n \text{ if } \frac{1}{n} \le x \le \frac{2}{n} & E_1 \\ 0 \text{ otherwise} & E_2 \end{cases}$

Hence $\lim_{n\to\infty} f_n(x) = f(x) = 0$ a.e $\Rightarrow \int_0^1 f(x)dx = 0$

Also $\int_0^1 f_n(x)dx = \int_{E_1} f_n(x)dx + \int_{E_2} f_n(x)dx$

$= \int_{1/n}^{2/n} -n\, dx + 0 = -1$

Thus $\underline{\lim} \int_0^1 f_n(x)dx = -1$

$$\Rightarrow \int_0^1 f(x)dx \nleq \underline{\lim} \int_0^1 f_n(x)dx$$

3.34. Theorem(Lebesgue Monotone Convergence theorem). Let $< f_n >$ be an increasing sequence of non negative measurable functions and let $f = \lim f_n$. Then

$$\int f = \lim \int f_n$$

Proof. By Fatou's Lemma we have

$$\int f \le \underline{\lim} \int f_n$$

But for each n we have $f_n \le f$, son $\int f_n \le \int f$. But this implies

$$\overline{lim} \int f \le \int f$$

Hence

$$\int f = lim \int f_n$$

3.35. Definition. A non-negative measurable functions f is called integrable over the measurable over the measurable set E if

$$\int_E f < \infty$$

3.36. Theorem. Let f and g be two non-negative measurable functions. If f is integrable over E and g(x)<f(x) on E, then g is also integrable on E, and

$$\int_E (f-g) = \int_E f - \int_E g$$

Proof. Since $\int_E f = \int_E (f-g) + \int_E g$

and the left hand side is finite, the term on the right must also be finite and so g is integrable.

3.37. Theorem. Let f be a non-negative function which is integrable over a set E. The given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ we have

$\int_A f < \varepsilon$

Proof. If $|f| \leq K$, then $\int_A f \leq \int_A K = KmA$

Set $\delta < \frac{\epsilon}{K}$ Then $\int_A f < K.\frac{\epsilon}{K} = \epsilon$.

Set $f_n(x) = f(x)$ if $f(x) \leq n$ and $f_n(x) = n$ otherwise. Then each f_n is bounded and f_n converges to f at each point. By the monotone convergence theorem there is an N such that $\int_E f_N > \int_E f - \frac{\epsilon}{2}$ *and* $\int_E (f - f_N) < \frac{\epsilon}{2}$.

Choose $\delta < \frac{\epsilon}{2N}$. If mA < δ , we have

$$\int_A f = \int_A (f - f_N) + \int_A f_N$$

$$< \int_E (f - f_N) + NmA$$

$$\left(since \int_A f_N \leq \int_A N = NmA\right)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \ .$$

3.38. The General Lebesgue Integral

We have already defined the positive part f^+ and negative part f^- of a function as

$$f^+ = max(f, 0)$$

$$\overline{f} = max(-f, 0)$$

Also it was shown that

$$f = f^+ - \overline{f}$$

$$|f| = f^+ + \overline{f}$$

With these notions in mind, we make the following definition.

3.39. Definition. A measurable function f is said to be integrable over E if f^+ and $\overline{f}$ are both integrable over E. In this case we define

$$\int_E f = \int_E f^+ - \int_E \overline{f}$$

3.40. Theorem. Let f and g be integrable over E. Then

(i) The function f+g is integrable over E and

$$\int_E (f+g) = \int_E f + \int_E g$$

(ii) If $f \leq g\ a.e.,$ then

$$\int_E f \leq \int_E g$$

(iii) If A and B are disjoint measurable sets contained in E, then

$$\int_{A\cup B} f = \int_A f + \int_B f$$

Proof. By definition, the function $f^+, \overline{f}, g^+, \overline{g}$ are all integrable. If $h = f+g$, then $h = (f^+ -, \overline{f}) + (g^+ - \overline{g})$ and hence $h = (f^+ + g^+) - (\overline{f} + \overline{g})$. Since $f^+ + g^+$ and $\overline{f} + \overline{g}$ are integrable therefore their difference is also integrable. Thus h is integrable.

We then have

$$\int_E h = \int_E [(f^+ + g^+) - (\overline{f} + \overline{g})]$$

$$= \int_E (f^+ + g^+) - \int_E (\overline{f} + \overline{g})$$

$$= \int_E f^+ + \int_E g^+ - \int_E \overline{f} - \int_E \overline{g}$$

$$= (\int_E f^+ - \int_E \overline{f}) + (\int_E g^+ - \int_E \overline{g})$$

That is, $\int_E (f+g) = \int_E f + \int_E g$

Proof of (ii) follows from part (i) and the fact that the integral of a non-negative integrable function is non-negative.

For (iii) we have $\int_{A\cup B} f = \int f_{\chi_{A\cup B}}$

$$= \int f\chi_A + \int f\chi_B = \int_A f + \int_B f$$

∗It should be noted that f+g is not defined at points where f = ∞ and g = −∞ and where f = −∞ and g = ∞ . However, the set of such points must have measure zero, since f and g are integrable. Hence the integrability and the value of $\int(f+g)$ is independent of the choice of values in these ambiguous cases.

3.41. Theorem. Let f be a measurable function over E. Then f is integrable over E iff $|f|$ is integrable over E. Moreover, if f is integrable, then

$$\left|\int_E f\right| = \int_E |f|$$

Proof. If f is integrable then both f^+ and f^- are integrable. But $|f| = f^+ + f^-$. Hence integrability of f^+ and f^- implies the integrability of $|f|$.

Moreover, if f is integrable, then since $f(x) \le |f(x)| = |f|(x)$, the property which states that if $f \le g$ a.e. , then $\int f \le \int g$ implies that

$$\int f \le \int |f| \qquad (i)$$

On the other hand since $-f(x) \le |f(x)|$, we have

$$-\int f \le \int |f| \qquad (ii) \quad .$$

From (i) and (ii)

Conversely, suppose f is measurable and suppose $|f|$ is integrable. Since

$$0 \le f^+(x) \le |f(x)|$$

It follows that f^+ is integrable. Similarly f^- is also integrable and hence f is integrable.

3.42. Lemma. Let f be integrable . Then given $\varepsilon > 0$ there exist $\delta > 0$ such that $\left|\int_A f\right| < \epsilon$ whenever A is measurable function f we have $= f^+ - f^-$. So by that we have proved already, given > 0 , there exist $\delta_1 > 0$ such that

$$\int_A f^+ < \frac{\varepsilon}{2}.$$

When mA<δ_1. Similarly there exists $\delta_2 > 0$ such that

$$\int_A f^- < \frac{\varepsilon}{2},$$

When mA<δ_2. Thus if mA < δ = min (δ_1, δ_2) , we have

$$\left|\int_A f\right| \le \int_A |f| = \int_A f^+ + \int_A f^- < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This completes the proof.

3.43. Theorem (**Lebesgue Dominated Convergence Theorem**) Let a sequence $< f_n >$, $n \varepsilon N$ of measurable functions be dominated by an integrable function g, that is

$$|f_n(x)| \leq g(x)$$

Holds for every $n \varepsilon N$ and every $x \varepsilon N$ and let $< f_n >$ converges pointwise to a function f, that is , f(x) = $\lim_{n\to\infty} f_n(x)$ for almost all x in E. Then

$$\int_E f = \lim_{n\to\infty} \int_E f_n$$

Proof. Since $|f_n| \leq g$ for every $n \varepsilon N$ and f(x) = $\lim_{n\to\infty} f_n(x)$, we have $|f| \leq g$. Hence f_n and f are integrable. The function $g - f_n$ is non-negative, therefore by Fatou's Lemma we have

$$\int_E g - \int_E f = \int_E (g - f) \leq \underline{lim} \int_E (g - f_n)$$

$$= \int_E g - \overline{lim} \int_E f_n$$

Whence $\int_E f \geq \overline{\lim} \int_E f_n$

Similarly considering $g + f_n$ we get

$$\int_E f \leq \overline{\lim} \int_E f_n$$

Consequently, we have $\int_E f = \lim \int_E f_n$